Discrete Element Structural Theory of Fluids

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A new type of structural element for discrete element idealization of fluids is investigated. With this structural element and the standard structural elements, a missile and liquid propellant can be idealized as one composite structure consisting of many elements. A theory for taking slosh into account is derived by establishing a slosh stiffness matrix for the fluid elements. This development enables all linear displacement modes of a missile and liquid propellant, including slosh modes and coupled slosh and structural modes, to be computed. Some of the modes, the number depending on the number of fluid elements, are zero frequency modes. This reflects a true liquid characteristic. The fluid elements can be compressible or incompressible. Slosh modes and frequencies are computed for several discrete element fluid idealizations.

Introduction

USING the fluid discrete element technique,[1] a missile and liquid propellant, as an example, may be idealized as one composite structure consisting of many elements. The resulting idealization would lead to mass and stiffness matrices that could become the basis for a structural vibration analysis.

A discrete element idealization of a fluid is not needed for static stress analysis, for it is a simple matter to get fluid pressure from hydraulic head at any point in a static fluid. However, vibration analysis of fluids in a flexible container is much more complicated, and the discrete element technique applied to fluids appears to have important applications.

A basic assumption for a fluid element is that of uniform internal pressure. Equilibrium of an element with respect to its internal pressure is maintained by forces acting at nodes in the usual fashion of discrete element procedure. The resulting fluid idealization is statically an unstable structure. This reflects a true fluid characteristic, as a distinguishing property of a fluid is its instability as a static structure.

This paper develops the procedure and rationale for treating a fluid as an array of structural elements. Stiffness matrices are derived for both compressible and incompressible fluid elements, with the implicit assumption of a displacement field. This derivation takes into account surface tension and gravitational forces. The matrix due to gravitational forces is the slosh stiffness matrix and this matrix is discussed in detail.

The columns of the stiffness matrix are formed by applying a unit displacement at one coordinate at a time and computing the resulting forces at the nodes. Widely used for solid structural elements, this method of forming a stiffness matrix emphasizes the structural characteristics of a fluid, although a method based on variational principles could be used for determining the equilibrium conditions at the nodes. The latter technique is explained in Ref. 2.

Recent discrete element approaches[3-5] use the differential equations of flow as the basic formulation. Here, the continuum fluid is not considered a structure. For example, a velocity field is utilized with local element velocities specified in terms of nodal velocity, rather than a displacement field as used for structural elements. The result for a fluid flow problem is a matrix formulation that is analogous to that for a structure.

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Formulation of the Fluid Structural Element

A fluid element in the form of a rectangular parallelopiped with internal pressure p would be subjected to forces at the nodes as shown in Fig. 1. These forces are perpendicular to an element face. If the fluid is incompressible, the constraint equation involving node deformations is

$$\delta_a/a + \delta_b/b + \delta_c/c = 0 \tag{1}$$

If adjacent elements are the same shape and size, the mass matrix is formed by lumping the mass included by the rectangular parallelopiped at each of the six nodes. There is just one mass degree of freedom at a node, and it is in the direction perpendicular to a node face.

The fluid in a cylinder of circular cross section could be idealized by an array of elements attached to each other at nodes as shown in Fig. 2. This array consists of two kinds of elements which are the cylindrical and triangular elements shown in more detail in Fig. 3. The forces acting on the elements corresponding to an internal pressure p act at the nodes as indicated. The forces for each element are, of course, in equilibrium.

If the fluid is incompressible, the constraint equations involving node deformations for the cylindrical and triangular elements are, respectively,

$$(R-r)h\cos\alpha(\delta_b+\delta_c) - 2hR\sin\alpha\cos\alpha\delta_R + 2hr\sin\alpha\cos\alpha\delta_r - [2r(R-r)\sin\alpha\cos\alpha + (R-r)^2\sin\alpha\cos\alpha]\delta_h = 0 \tag{2}$$

and

$$Rh\cos\alpha(\delta_b+\delta_c) - 2hR\sin\alpha\cos\alpha\delta_R - R^2\sin\alpha\cos\alpha\delta_h = 0 \tag{3}$$

The deformations are with reference to the nodes and, disregarding sign, in the direction of the forces at the nodes. Deformations δ_R, δ_r, and δ_h are arbitrarily positive and δ_b and δ_c negative when the corresponding dimension of an element is increased.

Fluid elements of other shapes can be formulated. Instead of the twelve triangular elements (Fig. 2), one element in the

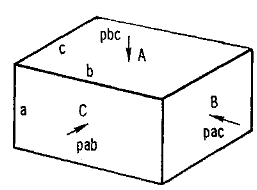

Fig. 1 Rectangular parallelopiped element.

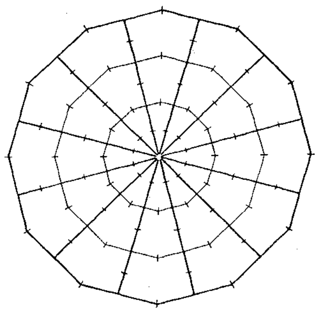

Fig. 2 Cross-sectional array of elements.

shape of a twelve sided regular polygon could be used. Boundary elements adjacent to a curved portion of a dome are necessary for practical application and these are shown in Fig. 4, along with some of the element forces. Further discussion of these elements is in Ref. 1.

Slosh Stiffness Matrix for an Incompressible Fluid

Consider fluid in a tube as shown in Fig. 5. When the fluid is displaced a distance q, the force at "0" for static equilibrium is

$$F = 2\rho A q \tag{4}$$

where ρ is the fluid weight density and A is the tube cross-sectional area.

The product $2\rho A$ can be interpreted as a spring constant with respect to q, in the usual manner of the displacement method. This concept is now used to derive the slosh stiffness matrices for several fluid idealizations in rigid containers. For simplicity, the fluid elements are square, planar elements that represent a fluid subjected to planar motion and are otherwise the same as the three-dimensional rectangular parallelopiped elements previously discussed.

The slosh stiffness matrix are given with respect to generalized coordinates. For each idealization, the mass matrix with respect to the generalized coordinates is also given. This mass matrix is derived from a diagonal matrix (with respect to the total coordinates) by a coordinate transformation operation. The transformation operation is discussed later. Elements of the diagonal matrix corresponding to surface and interior coordinates are W/g and $2W/g$, respectively, based on an element weight of $2W$.

A container with two fluid elements is shown in Fig. 6. From the constraint equations of incompressibility, coordinates q_1 and q_2 can be eliminated, leaving the generalized coordinate q_3. The slosh stiffness matrix with respect to q_3 is $2\rho A$, where A is a fluid element cross-sectional area. Note that the term $2\rho A$ is the same as obtained for the tube of fluid

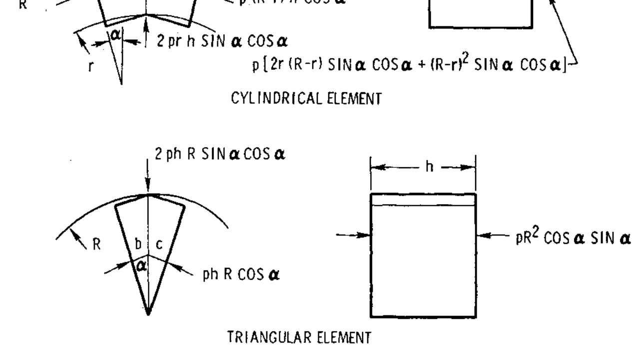

Fig. 3 Cylindrical and triangular elements.

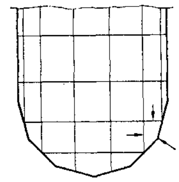

Fig. 4 Elements in a dome section.

in Fig. 5. The mass matrix with respect to q_3 is $4W/g$. The resulting slosh frequency is

$$p^2 = \rho g A/2W \tag{5}$$

A container with four fluid elements and the mass and slosh stiffness matrices are shown in Fig. 7. The generalized coordinates are q_3 and q_6. The resulting slosh frequency is

$$p^2 = 4\rho g A/7W \tag{6}$$

and the slosh mode is $q_3{:}q_6 = 1{:}\frac{1}{3}$.

The slosh stiffness matrix is determined in the usual manner. Namely, one coordinate at a time is given a unit displacement, and the resulting forces at the nodes for static equilibrium are computed. The dashed lines in Fig. 7 represent a unit displacement of coordinate q_3. The resulting forces at nodes 3 and 6 are each $2\rho A$. In this way a column of the stiffness matrix due to gravitational effects is generated. In the case of a fluid, the forces are due to hydraulic head in the manner of the tube of fluid. The forces are due to hydraulic head in the manner of the tube of fluid of Fig. 5.

The fluid idealization of Fig. 7 with the lower elements representing a fluid with a density n times the density of the upper elements or $n\rho$ is now considered. The mass and slosh stiffness matrices with respect to the generalized coordinates q_3 and q_6 are, respectively,

$$\begin{bmatrix} 4W/g & 2W/g \\ 2W/g & 4W(1+n)/g \end{bmatrix}$$

and

$$\begin{bmatrix} 2\rho A & 2\rho A \\ 2\rho A & 2n\rho A \end{bmatrix}$$

For $n = 6$, the slosh frequencies are

$$p^2 = \rho g A/3W \text{ and } p^2 = 5\rho g A/9W \tag{7}$$

The modes are $q_3{:}q_6 = 1{:}-\frac{1}{2}$ and $q_3{:}q_6 = 1{:}\frac{1}{4}$.

Note that increasing the density of the lower elements as compared to the upper elements results in two nonzero frequency modes instead of one. The two density idealization has the smaller frequency, and for the lower mode the horizontal fluid velocity changes sign as the boundary separating the fluid of two densities is crossed. These results are in qualitative agreement with those in Ref. 6 for a similar problem.

A container with six fluid elements and the mass and slosh stiffness matrices are shown in Fig. 8. The generalized co-

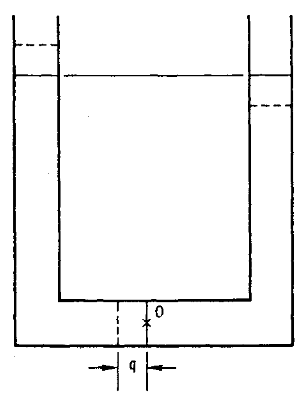

Fig. 5 Fluid in a tube.

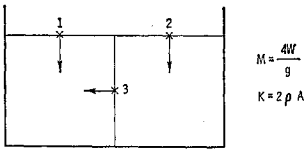

Fig. 6 Two fluid element array.

ordinates are q_3, q_6, and q_9. The slosh frequency is

$$p^2 = 15\rho gA/26W \tag{8}$$

and the slosh mode is $q_3{:}q_6{:}q_9 = 1{:}\frac{3}{11}{:}\frac{1}{11}$. There are two zero frequency modes.

A container with three fluid elements and the mass and slosh stiffness matrices are shown in Fig. 9. The generalized coordinates are q_4 and q_5. The slosh frequencies are

$$p^2 = \rho gA/3W \text{ and } p^2 = 3\rho gA/5W \tag{9}$$

The corresponding modes are $q_4{:}q_5 = 1{:}1$ and $q_4{:}q_5 = 1{:}-1$.

Lamb's equation[7] for lowest slosh frequency of fluid in a rectangular tank is

$$p^2 = (g\pi/b)\tanh(\pi h/b) \tag{10}$$

This formula for the aforementioned tank gives

$$p^2 = 0.81\ g/c \tag{11}$$

which compares to the frequency

$$p^2 = \rho gA/3W \text{ or } p^2 = 0.67\ g/c \tag{12}$$

from the discrete element model. This is good agreement for such a coarse discrete element model.

A slosh stiffness matrix can be formed with respect to the total coordinates. Consider a column of fluid elements of an array of elements in a container. The change in hydraulic head of a column of elements can be thought of as equivalent to a force acting at the surface node of the top element. This force is proportional to the coordinate vertical displacement. In this way, an element of the slosh stiffness matrix with respect to a surface coordinate is formed. All elements of the slosh stiffness matrix with respect to interior coordinates and all coupling elements are zero.

For the idealization of Fig. 9, the slosh stiffness matrix with respect to the total coordinates, q_1, q_2, q_3, q_4, q_5 is

$$K = \begin{bmatrix} \rho A & & & & \\ 0 & \rho A & & & \\ 0 & 0 & \rho A & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

and the mass matrix is

$$M = \begin{bmatrix} W/g & & & & \\ 0 & W/g & & & \\ 0 & 0 & W/g & & \\ 0 & 0 & 0 & 2W/g & \\ 0 & 0 & 0 & 0 & 2W/g \end{bmatrix} \tag{14}$$

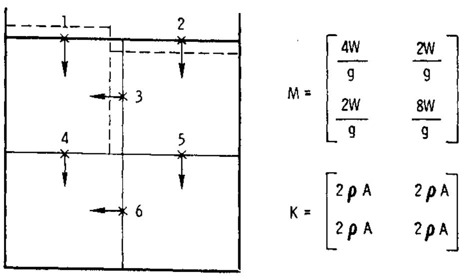

Fig. 7 Four fluid element array.

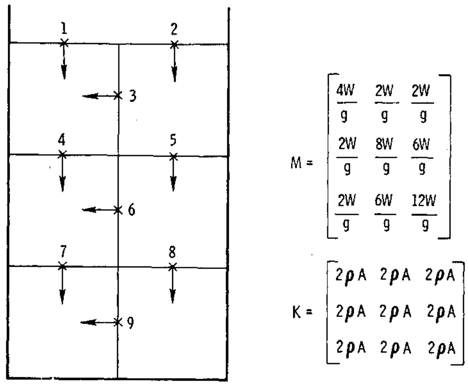

Fig. 8 Six fluid element array.

The transformation matrix relating the total coordinates to the generalized coordinates, q_4 and q_5, from the incompressibility constraint equations is

$$T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

The mass and stiffness matrices with respect to the generalized coordinates are, respectively, T^tMT and T^tKT which of course equal the matrices in Fig. 9.

Consider the array of twenty-four planar fluid elements in a rigid container shown in Fig. 10. The slosh frequencies and modes were found by first forming the mass and stiffness matrices with respect to the total coordinates and transforming to the generalized coordinates as just described. These matrices with respect to the generalized coordinates are M_q and K_q and each twenty by twenty.

There are fifteen zero frequency modes. The five nonzero frequencies obtained by a computer run along with frequencies calculated from Lamb's equation are tabulated in Table 1. Mode shapes, plotting only deflections of nodes 1–6, are shown in Fig. 11.

This problem was also solved by eliminating the zero frequency modes and forming reduced mass and stiffness matrices that contain only the nonzero frequency modes. The expression

$$F^tM_qq = 0 \tag{16}$$

was used to form the necessary constraint equations. The matrix of zero frequency modes F is shown in Fig. 12. The formulation for the reduced problem is discussed later.

For the examples of fluid arrays that have been discussed, the fluid was in a rigid container. The slosh stiffness matrix for fluid in a flexible container will now be derived for one example.

A flexible container with two fluid elements is shown in Fig. 13. Generalized coordinates are q_3, q_4, q_5, q_6, and q_7. Applying

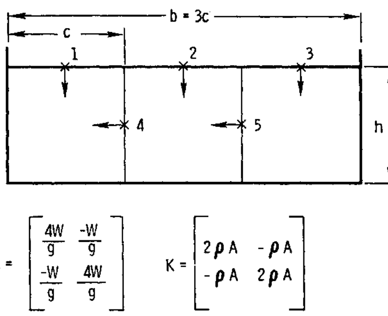

Fig. 9 Three fluid element array.

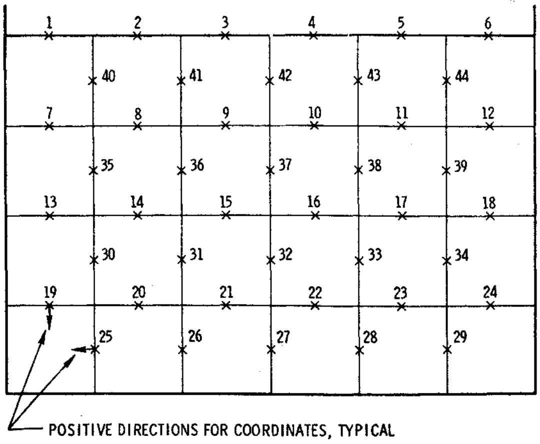

Fig. 10 Twenty-four fluid element array.

a unit displacement at one coordinate at a time, and computing the resulting forces required at all the generalized coordinates, the following slosh stiffness matrix is derived:

$$\begin{bmatrix} \rho A & & & & \\ -\rho A & 2\rho A & & & \\ 0 & -\rho A & \rho A & & \\ \rho A & -\rho A & 0 & 0 & \\ 0 & \rho A & -\rho A & 0 & 0 \end{bmatrix}$$

This matrix illustrates the character of the slosh stiffness matrix for any array of fluid elements in a flexible container. The total stiffness matrix of the fluid array and flexible container is obtained by adding the slosh stiffness matrix and the structure stiffness matrix of the container.

Matrix Formulation of the Reduced Problem for Incompressible Fluids

The original fluid mass matrix M_z and the stiffness matrix K_z correspond to the total coordinates

$$q_z = \begin{Bmatrix} q_c \\ \hline q_0 \\ \hline q_a \\ \hline q_b \end{Bmatrix} \tag{17}$$

where q_c are fluid coordinates eliminated by the constraint equations of incompressibility, q_0 are eliminated by the zero frequency mode elimination, q_a are the remaining fluid coordinates and the container coordinates, and q_b are structure coordinates.

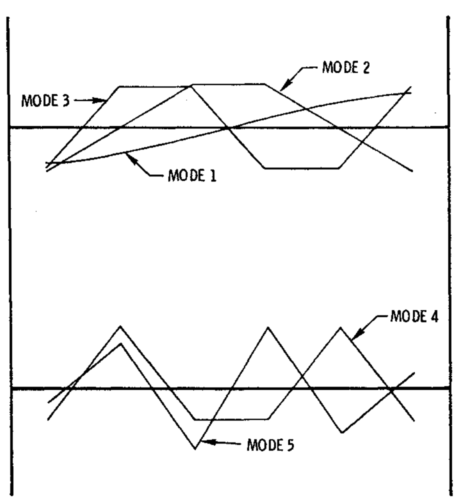

Fig. 11 Surface modes for the twenty-four element array.

Table 1 Frequency comparison

Mode	Frequencies—p^2 Discrete element	Lamb's equation
1	18.7	19.6
2	34.5	40.4
3	44.6	60.6
4	50.5	80.8
5	53.6	100.8

From the incompressible constraint equations

$$q_z = T\begin{Bmatrix} q_0 \\ \hline q_a \\ \hline q_b \end{Bmatrix} \tag{18}$$

If F is the matrix of zero frequency modes, then

$$F^tT^tM_zT\begin{Bmatrix} q_0 \\ \hline q_a \\ \hline q_b \end{Bmatrix} = 0 \tag{19}$$

Let

$$F^tT^tM_zT = [B \mid C] \tag{20}$$

Then

$$\begin{Bmatrix} q_0 \\ \hline q_a \\ \hline q_b \end{Bmatrix} = \begin{bmatrix} -B^{-1}C \\ \hline I \end{bmatrix}\begin{Bmatrix} q_a \\ \hline q_b \end{Bmatrix} \tag{21}$$

Hence

$$q_z = TD\begin{Bmatrix} q_a \\ \hline q_b \end{Bmatrix} \text{ where } D = \begin{bmatrix} -B^{-1}C \\ \hline I \end{bmatrix} \tag{22}$$

The reduced mass and stiffness matrices are with respect to coordinates q_a and q_b and are

$$M_r = D^tT^tM_zTD \tag{23}$$

and

$$K_r = D^tT^tK_zTD \tag{24}$$

Stiffness Matrix due to Fluid Surface Tension

The stiffness matrix due to surface tension is given for the three element array of Fig. 9. If S is the surface tension force per unit length and considering cubical elements, the stiffness with respect to the total coordinates, q_1, q_2, q_3, q_4 and q_5, is

$$\begin{bmatrix} S & & & & \\ -S & 2S & & & \\ 0 & -S & S & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{array}{c} 25\\26\\27\\28\\29\\30\\31\\32\\33\\34\\35\\36\\37\\38\\39\\40\\41\\42\\43\\44 \end{array} & \left[\begin{array}{ccccccccccccccc}
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\end{array}\right] \end{array}$$

Fig. 12 Zero frequency modal matrix.

This stiffness matrix is similar to that for a beam with a longitudinal force, while a three-dimension array of fluid elements and a plate with in-place forces have similar stiffness matrices.

Stiffness Matrices for a Compressible Fluid

The slosh stiffness matrices considering compressible fluid elements for two arrays discussed in the incompressible section will now be given, considering rigid containers for simplicity. For a compressible fluid, the slosh stiffness matrix is with respect to the total coordinates. The slosh stiffness matrices for the two arrays follow. Four fluid element array (Fig. 7):

$$\begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 2\rho A & & & \\ 0 & 0 & -\rho A & 0 & & \\ 0 & 0 & \rho A & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 2\rho A \end{bmatrix}$$

Six fluid element array (Fig. 8):

$$\begin{bmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & 2\rho A & & & & & & \\ 0 & 0 & -\rho A & 0 & & & & & \\ 0 & 0 & \rho A & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 2\rho A & & & \\ 0 & 0 & 0 & 0 & 0 & -\rho A & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \rho A & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\rho A \end{bmatrix}$$

The fluid compressibility stiffness matrix is given for the four fluid element array of Fig. 7. If E is the fluid bulk modulus, and t is the dimension of a cubical planar fluid element, the compressibility stiffness matrix is

$$\begin{bmatrix} Et & & & & & \\ 0 & Et & & & & \\ Et & -Et & 2Et & & & \\ -Et & 0 & -Et & 2Et & & \\ 0 & -Et & Et & 0 & 2Et & \\ 0 & 0 & 0 & Et & -Et & 2Et \end{bmatrix}$$

Concluding Remarks

A discrete element idealization of an incompressible fluid in a container, neglecting slosh, leads to a method for computing a fluid mass matrix with respect to the container coordinates. In this case, the fluid does not affect the over-all structure stiffness matrix. Some of the resulting vibration modes are zero frequency modes, the number depending on the number of fluid elements. This phenomenon reflects a true fluid property, since a real fluid has an infinite number of zero frequency modes.

On the other hand, the inclusion of slosh effects permits discrete element theory of fluids to be considered in a more general manner. When slosh is taken into account, slosh modes result and the number of zero frequency modes decreases.

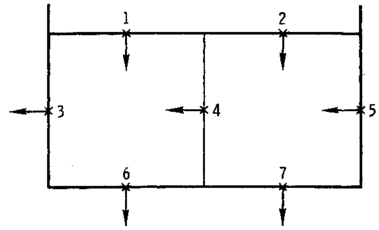

Fig. 13 Two fluid element array with flexible container.

Slosh is accounted for by deriving a stiffness matrix due to gravitational effects. The equations of motion based on the slosh stiffness matrix lead to slosh frequencies and modes for fluid in a rigid container. Stiffness matrices due to slosh and surface tension added to the structure stiffness matrix of a flexible container leads to equations of motion which give all modes and frequencies (slosh and coupled slosh and structural modes). For a compressible fluid, a fluid compressible stiffness matrix can be formed and included with the over-all structure stiffness matrix that takes into account fluid compressibility effects.

Again, for an incompressible fluid, neglecting slosh and surface tension, fluid effects can be taken into account by solving a reduced problem that does not include surface or interior fluid coordinates. Slosh can be accounted for with the number of fluid coordinates in the reduced problem proportional to the fluid container cross-sectional area and not to the fluid volume. (This is an important point because the size of a problem is limited by computer capability and the economics of computer run time.)

Instead of the modal method, dynamic solution of a structural idealization of compressible fluids elements can be based on a system of recursive equations. Such a method is explained in Ref. 8.

References

[1] Hunt, D. A., "Discrete Element Idealization of an Incompressible Liquid for Vibration Analysis," *AIAA Journal,* Vol. 8, No. 6, June 1970, pp. 1001–1004.

[2] Zienkiewicz, O. C. and Chenug, Y. K., "Finite Element Method in Structural and Continuum Mechanics," McGraw-Hill, London, 1967, pp. 18–25 and 148–169.

[3] Thompson, E. G., Mack, L. R., and Lin, F. S., "Finite Element Method for Incompressible Slow Viscous Flow with a Free Surface," *Developments in Mechanics, Vol. 5, Proceedings of the 11th Midwestern Mechanics Conference,* Iowa State University Press, 1969, pp. 93–111.

[4] Oden, J. T. and Somogyi, D., "Finite Element Applications in Fluid Dynamics," *Journal of the Engineering Mechanics Division, Proceedings of the ASCE,* Vol. 95, EM3, June 1969, pp. 821–826.

[5] Luk, C. H., "Finite Element Analysis for Liquid Sloshing Problems," ASRL TR 144-3, Aug. 1969, MIT.

[6] Lamb, H., *Hydrodynamics,* 6th ed., Dover, New York, 1945, pp. 370–372.

[7] Abramson, H. N., *The Dynamic Behavior of Liquids in Moving Containers,* NASA SP-106, 1966, p. 17.

[8] Ang, A. H.-S., "Numerical Approach for Wave Motions in Nonlinear Solid Media," Matrix Methods in Structural Analysis, TR-66-80, Nov. 1966, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, pp. 753–778.